LEZIONE 16

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$

$$\Gamma\left(\frac{1}{2}\right) \quad \overset{?}{\underset{0}{}}$$

$$\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = 1$$

u

$\sqrt{u}$

$$\Gamma(x+1) = x\,\Gamma(x)$$

$$\forall n \in \mathbb{N} \quad \Gamma(n+1) = n!$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\, dt$$

$$= \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\, dt \qquad \sqrt{t} = u \qquad t = u^2$$

$$dt = 2u\, du$$

$$= \int_0^{\infty} \frac{e^{-u^2}}{u}\, 2u\, du = 2\int_0^{\infty} e^{-u^2}\, du = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

Se m, n siano interi:

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}$$ si dice coeff. Bin.

Se il numeratore non è intero $m = \alpha \in \mathbb{R}^+$

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\cdot n!}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$ si chiama integrale Euleriano di seconda specie

L'integrale euleriano di prima specie, integrale Beta,

$$B(x,y) = \int_0^{1} t^{x-1} (1-t)^{y-1}\, dt$$

l'integrale converge per $1-x < 1$ $\quad 1-y < 1$

cioè converge per $x, y > 0$

(1) $$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$$

(2) $$B(x,y) = 2 \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \, d\theta \qquad t = \cos^2\theta$$

(3) $$B(x,y) = \int_0^{\infty} \frac{s^{x-1}}{(1+s)^{x+y}} \, ds \qquad s = \frac{t}{1-t}$$

TEOREMA $$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

Dim. Calcolo il prodotto $\Gamma(x)\,\Gamma(y)$ e poi applico Fubini

$$\Gamma(x)\Gamma(y) = \int_0^{\infty} e^{-t} t^{x-1} dt \int_0^{+\infty} e^{-s} s^{y-1} ds$$

$t = u^2 \qquad s = v^2 \qquad (u^2)^{x-1} \qquad (v^2)^{2y-1}$

$dt = 2u\,du \qquad ds = 2v\,dv$

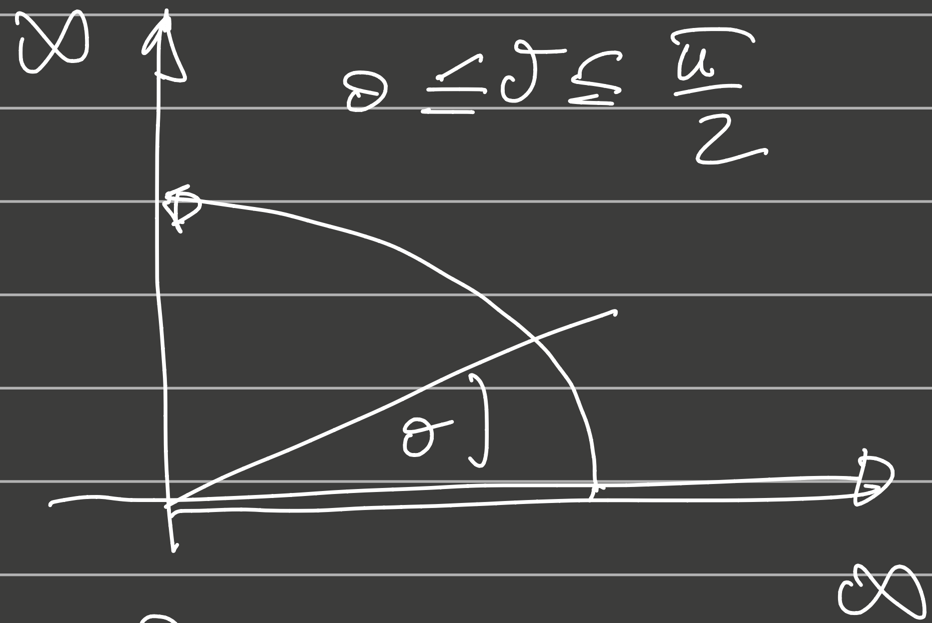
$$= 2\int_0^{\infty} e^{-u^2} u^{2x-2} \cdot u\,du \cdot 2\int_0^{\infty} e^{-v^2} v^{2y-2} v\,dv$$

$$= 4\int_0^{\infty} e^{-u^2} u^{2x-1} du \int_0^{\infty} e^{-v^2} v^{2y-1} dv$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)} u^{2x-1} v^{2y-1} dv\,du$$

qui uso le coordinate polari $\qquad x = \rho\cos\theta$, $y = \rho\sin\theta$

$0 \le \theta \le \frac{\pi}{2}$



$$4\int_0^{\pi/2}\int_0^{\infty} \rho^{2x-1}(\cos\theta)^{2x-1}\cdot \rho^{2y-1}(\sin\theta)^{2y-1}\,\rho\, e^{-\rho^2}\,d\rho\, d\theta =$$

jacobiano

$$2\int_0^{\infty} \rho^{2x-1+2y-1+1}\, e^{-\rho^2}\,d\rho \cdot 2\int_0^{\infty} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta$$

$$= 2\int_0^{\infty} \rho^{2(x+y)-1}\, e^{-\rho^2}\,d\rho \cdot B(x,y)$$

In conclusione ricordiamo che il cambio di variabile $t=u^2$ ci ha permesso di ottenere una rappresentazione alternativa di $\Gamma(x)$

$$\Gamma(x) = 2\int_0^{\infty} u^{2x-1} e^{-u^2}\, du$$

Ora

$$2\int_0^{\infty} \rho^{2(x+y)-1} e^{-\rho^2}\, d\rho = \Gamma(x+y)$$

Conclusione

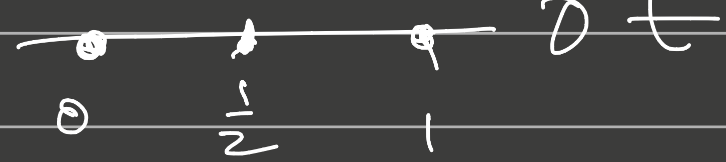
$\Gamma(x)\Gamma(y) = \Gamma(x+y)\, B(x,y)$ che è la tesi!

Dalla funzione Beta si ritrova il fatto che $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \left(\Gamma\left(\tfrac{1}{2}\right)\right)^2$$

Q 2a $\quad B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{1}{2}-1} \, dt$

$$= \int_0^1 \frac{dt}{\sqrt{t(1-t)}}$$



$s = t - \frac{1}{2}$

$t = 0 \Rightarrow s = -\frac{1}{2}$

$t = 1 \Rightarrow s = \frac{1}{2}$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{ds}{\sqrt{\frac{1}{4} - s^2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\sqrt{1-4s^2}} \, ds$$

$ds = dt$

$t(1-t) \qquad t = s + \frac{1}{2}$

$\left(s+\frac{1}{2}\right)\left(1 - s - \frac{1}{2}\right)$

$\left(s+\frac{1}{2}\right)\left(-s + \frac{1}{2}\right)$

$\left(\frac{1}{2} - s\right)\left(\frac{1}{2} + s\right)$

$$\sqrt{\frac{1}{4} - s^2} = \sqrt{\frac{1}{4}\left(1 - \frac{s^2}{\frac{1}{4}}\right)}$$

$$= \frac{1}{2}\sqrt{1 - (2s)^2}$$

$$\int_0^1 \frac{dt}{\sqrt{t(1-t)}} = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{ds}{\sqrt{1-(2s)^2}} = \int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}}$$

$$2s = y \qquad s = \frac{1}{2}y \qquad ds = \frac{1}{2}dy \qquad = 2\arcsin 1 = \pi$$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi = \left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

$$\Longrightarrow \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Formula di riflessione

$$I = \int_0^1 t^{-3/4}(1-t)^{1/4}\, dt = \mu(A)$$

$\mu(A)$ con $A = \{ (x,y) \in \mathbb{R}^2 : \quad x^4 + y^4 \leq 1 \}$

$y-1=\frac{1}{4} \Rightarrow y=\frac{5}{4}$ $\quad x-1=-\frac{3}{4} \Rightarrow x=\frac{1}{4}$

$$\int_0^1 t^{-\frac{3}{4}}(1-t)^{\frac{1}{4}}\,dt = B\left(\frac{1}{4},\frac{5}{4}\right)$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{6}{4}\right)}$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$1+\frac{1}{2}$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}=\frac{\frac{1}{4}\Gamma^2\left(\frac{1}{4}\right)}{\frac{\sqrt{\pi}}{2}}$$

$$=\frac{2}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{\pi}}$$

Formula di Riflessione di Eulero

$\forall x \in]0,1[\quad \Gamma(x)\,\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$

$$\int_0^{+\infty} \frac{dx}{1+x^4}$$

$1+x^4 = p(x) \cdot q(x)$

con p, q pol di secondo grado con discriminante negativo

$$\frac{ax+b}{p} + \frac{cx+d}{q} = \frac{1}{1+x^4}$$

$1+x^4 = \dfrac{1}{t} \qquad 4x^3\,dx = -\dfrac{dt}{t^2}$

$\left(\dfrac{1}{t} - 1\right)^{1/4} = x \qquad x = \left(\dfrac{1-t}{t}\right)^{1/4}$

$$dx = \frac{1}{4}\left(\frac{1-t}{t}\right)^{\frac{1}{4}-1} \cdot \frac{dt}{t^2}$$

$$= -\frac{1}{4}(1-t)^{\frac{1}{4}-1}\, t^{\left(1-\frac{1}{4}\right)-2} = -\frac{1}{4}(1-t)^{\frac{1}{4}-1}\, t^{-\frac{5}{4}}$$

$$-\frac{1}{4}\int_1^0 t\,(1-t)^{\frac{1}{4}-1}\, t^{-\frac{5}{4}}\, dt = \frac{1}{4}\int_0^1 (1-t)^{\frac{1}{4}-1}\, t^{1-\frac{5}{4}}\, dt$$

$$= \frac{1}{4}\int_0^1 (1-t)^{\frac{1}{4}-1}\, t^{-\frac{1}{4}}\, dt \qquad x-1 = -\frac{1}{4}$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} = \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}\,\pi}{4}$$

$$\int_0^{\pi/2} \sqrt{\sin x}\, dx = \int_0^{\pi/2} (\sin x)^{\frac{1}{2}} (\cos x)^0\, dx$$

è un integrale $B\left(\frac{1}{2}, -\frac{1}{2}\right)$ ... $2y-1 =$

Terza Parte del Corso

- Trasformata di Fourier

(servirà per trasformare un'equazione alle derivate parziali in un'equazione differenziale ordinaria)

<u>Def</u> Se $f \in \mathcal{L}(\mathbb{R})$ integrale finito su tutta la retta la sua trasformata di Fourier è una funzione $\mathcal{F}f$, $\hat{f}$ così definita:

$$(\mathcal{F}f)(s) = \hat{f}(s) := \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} \cos(2\pi s t)\, f(t) - i \int_{-\infty}^{+\infty} \sin(2\pi s t)\, f(t)\,dt$$

Un valore della trasformata si ottiene facilmente se prendo $s = 0$

$$(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(t)\,dt$$

Purtroppo la TdiF non è definita nello stesso modo nelle diverse discipline

$$\sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t)\, e^{ibst}\, dt$$

per noi $b = -2\pi$ $\quad a = 1$

$$\mathcal{F} f(s) = \frac{1}{A} \int_{-\infty}^{\infty} e^{iBst} f(t)\, dt$$

OSSERVAZIONE se $f(t) = e^{-\pi t^2}$

$$\hat{f}(s) = e^{-\pi s^2}$$

$$f(t) = e^{-\pi t^2} \qquad \hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t}\, e^{-\pi t^2}\, dt$$

derivo rispetto a s

$$\frac{d}{ds}\hat{f}(s) = \int_{-\infty}^{\infty} \underbrace{-2\pi i t}\; e^{-2\pi i s t}\; \underbrace{e^{-\pi t^2}}\; dt$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i s t} \left(i\, e^{-\pi t^2} \right)' dt$$

$$\underbrace{\left[i\, e^{-\pi t^2} \cdot e^{-2\pi i s t} \right]_{t=-\infty}^{t=\infty}}_{=\,0} - \int_{-\infty}^{\infty} i\, e^{-\pi t^2} \left(-2\pi i s \right) e^{-2\pi i s t}\, dt$$

$$= -2\pi s \int_{-\infty}^{\infty} e^{-2\pi i s t}\, e^{-\pi t^2}\, dt$$

In definitiva ho provato che

$$\frac{d}{ds}\hat{f}(s) = -2\pi s\,\hat{f}(s)$$

quindi $\hat{f}(s) = \hat{f}(0)\,e^{-\pi s^2}$

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = 1$$

$$\hat{f}(s) = e^{-\pi s^2}$$